Fibrations IV

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1 Pointed Fibrations

The theory of fibrations may also be set up in the pointed category. The theory is more or less analogous so we will be scarce with details. However there are important issues which need to be addressed. For instance, in contrast to the story for cofibrations there are actually more unpointed fibrations than pointed fibrations. While the idea of keeping track of basepoints may seem unnecessary, it will of course be essential when we come to making sense of long fiber sequences.

Proposition 1.1 *A pointed map $p : E \to B$ is said to satisfy the* ***based homotopy lifting property*** *(bHLP) with respect to a based space X if whenever given the solid part of a strictly commutative diagram in Top_* of the form*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \wedge I_+ \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd} \tag{1.1}$$

the dotted arrow can be completed so as to make the whole diagram commute. The map p is said to be a ***pointed fibration*** *if it satisfies the bHLP with respect to all based spaces.* $\square$

We will also say that a pointed fibration is based, or that it is a fibration in Top_*. By unpointed, unbased, or free fibrations we will understand the objects discussed in previous lectures, and we shall also refer to these objects as fibrations in Top. Notice that if $p : E \to B$ is a pointed fibration, then it always has a well-defined **fibre** $F = p^{-1}(*) \subseteq E$. We call a sequence of pointed spaces and maps

$$F \xrightarrow{i} E \xrightarrow{p} B \tag{1.2}$$

a **strict fibration sequence** if p is a pointed fibration and i is the inclusion of the fibre of p.

Proposition 1.2 *Let $p : E \to B$ be a pointed map. If p is a fibration in Top_*, then p is a fibration in Top.*

Proof Let M be an unbased space and suppose given a pair (f, H) consisting of free maps $f : M \to E$ and $H : M \times I \to B$ satisfying $pf = H_0$. Add a disjoint basepoint to M and consider the following diagram in Top_*.

$$\begin{tikzcd} M_+ \arrow[rr, "f_+"] \arrow[d, "in_0"'] & & E \arrow[d, "p"] \\ (M \times I)_+ \cong M_+ \wedge I_+ \arrow[urr, dashed, "\widetilde{H}"] \arrow[rr, "H_+"] & & B \end{tikzcd} \tag{1.3}$$

where f_+, H_+ are the obvious extensions to basepoint preserving maps. Since p is a pointed fibration, the dotted arrow can be filled in so as to make the diagram commute. Then the restriction $\widetilde{H}|_{M \times I}$ solves the free homotopy lifting problem posed by (f, H). We conclude that p is a fibration in Top. ∎

A consequence of the proposition is that the same comments with regards to surjectivity apply to both pointed and unpointed fibrations. Namely that a pointed fibration need not be surjective but does surject onto any path component which meets its image. Clearly this includes the path-component containing the basepoint. This can be a bit misleading, since a pointed fibration $p : E \to B$ has a fibre $F = p^{-1}(*) \subseteq E$, but does not in general have a fibre over an arbitrary point $b \in B$ (since the inverse image may be empty). It is always possible to restrict p to the path-component of B containing the basepoint, but of course non-connected base spaces do arise in practice.

Example 1.1 For any space X the unique map $\emptyset \to X$ is an unpointed fibration which is not even a pointed map. $\square$

Example 1.2 Let $E = 0 \times I \cup I \times 0 \subseteq \mathbb{R}^2$ and $B = I$ with $p : E \to B$ the projection onto the first factor. Then p is not a fibration in Top. For example, in Top the diagram

$$\begin{tikzcd} * \arrow[r, "{(0,1)}"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ I \arrow[r, "{t \mapsto (t,0)}"] & B \end{tikzcd} \tag{1.4}$$

admits no diagonal. Since E, B are CW complexes, p cannot be a pointed fibration for any choice of basepoint for E.

On the other hand, if E is given the basepoint $(0, 0)$, then $p : E \to B$ is has the bHLP with respect to all path-connected based spaces. $\square$

Example 1.3 Let $\mathcal{S}$ be the Sierpinski space. Then the evaluation $e_0 : \mathcal{S}^I \to \mathcal{S}$ is an unpointed fibration. It is shown in [1] that e_0 is not regular. i.e. it is not always possible to lift constant homotopies through e_0 as constant homotopies. Consequently if $\mathcal{S}$ is based at its open point u, and $\mathcal{S}^I$ is based at the constant path at u, then e_0 is not a pointed fibration. Note that the inclusion $u \hookrightarrow \mathcal{S}$ is a non-closed cofibration. Thus below the requirements of well-pointedness over almost well-pointedness is essential. $\square$

Putting Proposition 1.2 and the last example together we see that there are strictly more pointed fibrations than unpointed. Nevertheless, free fibrations will always be well-behaved if we restrict to well-pointed spaces.

Lemma 1.3 *Let $p : E \to B$ be a free fibration. Then p satisfies the based homotopy lifting property with respect to any well-pointed space.*

Proof Assume given a homotopy lifting problem in Top_*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \wedge I_+ \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd} \tag{1.5}$$

in which X is well-pointed. By Proposition 1.2 p is an unpointed fibration, so we can consider (1.5) to be a diagram of unpointed spaces. Since X is well-pointed the inclusion $X \hookrightarrow X \wedge I_+$ is a free cofibration (cf. *Cofiber Sequences* Th. 2.2) and also a homotopy equivalence. Thus we may apply the fundamental lifting property to complete the dotted arrow in (1.5) so as to make the whole diagram commute. But this arrow is necessarily a basepoint preserving map, and hence is also a solution to the problem in Top_*. ∎

The formal properties of pointed and unpointed fibrations are identical irrespective of any special properties of the basepoints. For example, the following statements all follow from purely diagrammatic arguments.

Proposition 1.4 *The following statements hold.*

1) *If $p_1 : E_1 \to E_2$ and $p_2 : E_2 \to E_3$ are pointed fibrations, then so is $p_2 p_1 : E_1 \to E_3$.*

2) *If $p_i : E_i \to B_i$, $i = 1, 2$, are pointed fibrations, then so is the cartesian product $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$.* ∎

The next proposition is also formal and is dual to a similar statement made regarding cofibrations. We record it specially since we will need it in the sequel.

Proposition 1.5 *Assume that the square*

$$\begin{tikzcd} f^*E \arrow[r, "\widehat{f}"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{1.6}$$

is a pullback in Top_* *and that* p *is a pointed fibration. Then the pullback map* $q : f^*E \to A$ *is a pointed fibration, and the covering map* $\widehat{f} : f^*E \to E$ *induces a homeomorphism between the fibres*

$$q^{-1}(*) \xrightarrow{\cong} p^{-1}(*). \tag{1.7}$$

Proof We prove only the last part of the statement, for which we can assume that

$$f^*E = \{(a, e) \mid f(a) = p(e)\} \tag{1.8}$$

is the canonical pullback, topologised as a subspace of $A \times E$. Then q is the projection onto the first factor and

$$q^{-1}(*) = \{(*, e) \mid f(*) = * = p(e)\} = * \times p^{-1}(*). \quad \blacksquare \tag{1.9}$$

We end this section with a simple but important result. Although the statement will be greatly generalised in the sequel, this first step is fundamental.

Proposition 1.6 *Let*

$$F \xrightarrow{i} E \xrightarrow{p} B \tag{1.10}$$

be a strict fibration sequence. Then for any pointed space K*, the sequence*

$$[K, F] \xrightarrow{i_*} [K, E] \xrightarrow{p_*} [K, B] \tag{1.11}$$

is exact in Set_**.*

Proof We work in Top_*. It is sufficient to show that if $f : K \to E$ is a map such that pf is null homotopic, then f factors through i up to homotopy. So choose a homotopy $H : pf \simeq *$ and apply the based HLP to the diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ K \wedge I_+ \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd} \tag{1.12}$$

to get a homotopy $\widetilde{H}$ with $\widetilde{H}_0 = f$ and $p\widetilde{H}_1 = *$. Let $f' : K \to F$ be the corestriction of $\widetilde{H}_1$. Then $\widetilde{H}$ is a homotopy $f \simeq if'$. $\blacksquare$

1.1 The Pointed Mapping Path Space

Recall that the mapping path space W_f of an arbitrary map $f : X \to Y$ was defined in *Fibrations I* by the pullback square

$$\begin{tikzcd} W_f \arrow[r, "q_f"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\ulcorner"] & Y^I \arrow[d, "e_0"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{1.13}$$

Notice that if X, Y are based spaces and f is basepoint preserving, then (2.1) is a diagram of based spaces and is a pullback in Top_*. Here $Y^I = C(I, Y)$ is the space of unbased maps $I \to Y$ in the compact-open topology and is based at the constant map c_* at the basepoint of Y. In this case the pullback space

$$W_f \cong \{(x, l) \in X \times Y^I \mid f(x) = l(0)\} \tag{1.14}$$

has the canonical basepoint $(*, c_*)$ which is preserved by the maps $\pi_f : W_f \to X$ and $q_f : W_f \to Y^I$.

Recall that there is a map $\widetilde{f} : X^I \to W_f$ given by $\widetilde{f}(l) = (l(0), fl)$ and that a lifting function for f is a map $\lambda : W_f \to X^I$ which is a section of $\widetilde{f}$. We say that λ is a **pointed lifting function** if it is a pointed map.

Lemma 1.7 *A pointed map $f : X \to Y$ is fibration in Top_* if and only if it has a pointed lifting function.*

Proof The proof is purely formal and is identical to the unpointed case. ∎

When is a free fibration a pointed fibration? A sufficient condition is given in the next proposition.

Proposition 1.8 *Let $f : X \to Y$ be a map in Top which is a free fibration. Assume that Y has a basepoint y_0 which is the zero set of some map $\varphi : Y \to I$. Then f becomes a pointed fibration when X is given any basepoint $x_0 \in f^{-1}(*)$.*

Proof We produce a pointed lifting function for f. Begin by letting $\phi : Y^I \to I$ be the map

$$\phi(l) = \max_{t \in I}\{\varphi(l(t))\}. \tag{1.15}$$

Notice that $\phi^{-1}(0) = c_{y_0}$. Use this to define $\Phi : Y^I \to Y^I$ by setting

$$\Phi(l)(t) = \begin{cases} l(t/\phi(l)) & t < \phi(l) \\ l(t) & \phi(l) \le t \le 1. \end{cases} \tag{1.16}$$

Now, since f is a fibration it admits a lifting function $\lambda : W_f \to X^I$. Let $\widetilde{\lambda} : W_f \to X^I$ be the map

$$\widetilde{\lambda}(x, l)(t) = \lambda(x, \Phi(l))(\phi(l) \cdot t). \tag{1.17}$$

Then because λ is a lifting function we have for all $(x,t) \in W_f$ that

$$\widetilde{\lambda}(x,l)(t) = \lambda(x,\Phi(l))(0) = x \tag{1.18}$$

and

$$(f \circ \widetilde{\lambda}(x,l))(t) = (f \circ \lambda(x,\Phi(l))(\phi(l)\cdot t) = \Phi(l)(\phi(l)\cdot t) = l(t) \tag{1.19}$$

which shows that $\widetilde{\lambda}$ is also a lifting function for f. On the other hand, since $\widetilde{\phi}(c_*) = 0$ we have $\Phi(c_{y_0}) = c_{y_0}$, which implies that for any $x_0 \in f^{-1}(y_0)$ it holds that

$$\widetilde{\lambda}(x_0, c_{y_0})(t) = \lambda(x_0, c_{y_0})(0). \tag{1.20}$$

Together with (1.18) this shows that $\widetilde{\lambda}(x_0, c_{y_0}) \in X^I$ is the constant path at x_0. In particular $\widetilde{\lambda}$ is a pointed lifting function for f when X is equipped with x_0 as basepoint. ∎

Corollary 1.9 *Let $f : X \to Y$ be a pointed map to a well-pointed space Y. Then f is a pointed fibration if and only if it is a free fibration.*

Proof The forwards implication is Proposition 1.2. To prove the backwards implication choose a Strøm structure (φ, H) for f with $\varphi^{-1}(0) = \{*\}$ and apply Corollary 1.8. ∎

Corollary 1.10 *If Y is well-pointed, then the evaluation maps*

$$e_0 : Y^I \to Y, \qquad e_{0,1} : Y^I \to Y \times Y \tag{1.21}$$

are pointed fibrations. ∎

2 Replacing a Map With a Fibration

Let $f : X \to Y$ be a map and form its mapping path space W_f as the pullback

$$\begin{tikzcd} W_f \arrow[r, "q_f"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y^I \arrow[d, "e_0"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.1}$$

The evaluation e_0 is a fibration and a homotopy equivalence and in particular is shrinkable (cf. *Fibrations II* Corollary 4.5). Our favoured choice of section is the map $c_\bullet : Y \to Y^I$ which sends a point $y \in Y$ to the constant path c_y at that point. The implication is that the pullback map $\pi_f : W_f \to X$ is a both a fibration and a homotopy equivalence, so is shrinkable over X. The section in this case is the map

$$s_f : X \to W_f, \qquad x \mapsto (x, c_{f(x)}) \tag{2.2}$$

and the homotopy $F : s_f\pi_f \simeq id_{W_f}$ is given by

$$F_s(x,l) = (x, l^{[s]}) \tag{2.3}$$

where $l^{[s]} : I \to Y$ is the path

$$l^{[s]}(t) = l\,(s \cdot t)\,. \tag{2.4}$$

Now define a map $p_f : W_f \to Y$ as the composite

$$p_f : W_f \xrightarrow{q_f} Y^I \xrightarrow{e_1} Y, \qquad (x, l) \mapsto l(1). \tag{2.5}$$

Then this fits into a strictly commutative diagram

$$\begin{tikzcd} X \arrow[rr, "s_f"] \arrow[dr, "f"'] & & W_f \arrow[dl, "p_f"] \\ & Y. & \end{tikzcd} \tag{2.6}$$

Proposition 2.1 *The map $p_f : W_f \to Y$ is a fibration.*

Proof The idea is to 'pull back' a lifting function for the fibration $e_1 : Y^I \to Y$ to get one for p_f. Here are the details. Letting

$$W_f \times_Y Y^I = \{((x, l), m) \in X \times Y^I \times Y^I \mid f(x) = l(0),\ l(1) = m(0)\} \tag{2.7}$$

be the pullback of $W_f \xrightarrow{p_f} Y \xleftarrow{e_0} Y^I$ we need to define a map $\lambda : W_f \times_Y Y^I \to W_f^I$ satisfying

$$\lambda((x, l), m)(0) = (x, l), \qquad (p_{f*} \circ \lambda((x, l), m))(t) = p_{f*}(\lambda((x, l), m)(t)) = m(t). \tag{2.8}$$

To define λ we need to introduce some notation for a parametrised concatenation of paths. For two composable paths $l, m : I \to Y$ with $l(1) = m(0)$ and $s \in I$ we let $l +_{[s]} m : I \to Y$ be the path

$$(l +_{[s]} m)(t) = \begin{cases} l((1+s) \cdot t) & t \leq \frac{1}{1+s} \\ m((1+s) \cdot t - 1) & t \geq \frac{1}{1+s} \end{cases} \tag{2.9}$$

Notice that

$$(l +_{[0]} m)(t) = l(t), \qquad (l +_{[1]} m)(t) = (l + m)(t) \tag{2.10}$$

and

$$(l +_{[s]} m)(0) = l(0), \qquad (l +_{[s]} m)(1) = m(s) \tag{2.11}$$

Now define λ by setting

$$\lambda((x, l), m)(t) = (x, l +_{[t]} m), \qquad ((x, l), m) \in W_f \times_Y Y^I. \tag{2.12}$$

This is well defined by (2.11). The first equation of (2.12) is clearly satisfied, and to check the second we have

$$(p_{f*} \circ \lambda((x, l), m))(t) = p_{f*}((x, l +_{[t]} m)) = (l +_{[t]} m)(1) = m(t). \tag{2.13}$$

Thus λ is a lifting function for p_f. ∎

Thus in the diagram (2.6) we have succeeded in factoring f as a homotopy equivalence followed by a fibration. We say that $p_f : W_f \to Y$ is the result of **converting f into a fibration**.

Notice that if X, Y are based and the map f is basepoint preserving, then so is the section s_f and the homotopy (2.3) is one of pointed maps. Furthermore, in this case the lifting function produced in equation (2.12) preserves basepoints. Thus we are free to apply Lemma 1.7.

Corollary 2.2 *If $f : X \to Y$ is a based map, then $s_f : W_f \to Y$ is both a pointed and unpointed fibration.* ∎

In what sense, if any, is the construction of this section unique? Assume that we have found a space V and another factorisation of f as a homotopy equivalence $t : X \xrightarrow{\simeq} V$ followed by a fibration $q : V \to Y$. Then we have a commutative diagram

$$\begin{tikzcd} V \arrow[dr, "q"'] & X \arrow[l, "t"'] \arrow[r, "s_f"] \arrow[d, "f"] & W_f \arrow[dl, "p_f"] \\ & Y & \end{tikzcd} \tag{2.14}$$

and the question is how to relate V and W_f. If $u : V \to X$ is a homotopy inverse to t, then

$$p_f(s_f u) = fu = q(tu) \simeq q \tag{2.15}$$

so we can use the fact that p_f is a fibration to homotope $s_f u : V \to W_f$ to a map $v : V \to W_f$ satisfying $p_f v = q$. Then v is a map over Y and an ordinary homotopy equivalence. Applying *Fibrations II* Th. 4.4 we see that v is a homotopy equivalence over Y.

Proposition 2.3 *Fix a map $f : X \to Y$. Then up to homotopy equivalence over Y, the map $p_f : W_f \to Y$ is the unique way to replace f by a pointwise equivalent fibration. In particular, if f is already a fibration, then X is homotopy equivalent to W_f in Top/Y.* ∎

Example 2.1

1) *If $f : * \to Y$, then $W_f \cong P_0 Y = \{l : I \to X \mid l(0) = *\}$ and p_f is the map $p_f(l) = l(1)$.*

2) *If $f = * : X \to Y$ is the constant map, then $W_f \cong X \times P_0 Y$. In particular, if $f : X \to *$, then $W_f \cong X$.*

3) *If $f = id_X : X \xrightarrow{=} X$, then $W_f \cong X^I$ and p_f is the evaluation map $e_1 : l \mapsto l(1)$.*

4) *If $f : X \to Y$ and $f' : X' \to Y'$ are given, then there is a canonical homeomorphism $W_{f\times f'} \cong W_f \times W_{f'}$. As a special case, if $f = pr_X : X \times Y \to X$, then $W_f \cong X^I \times Y$.* □

2.1 Dependence on f

Recall the Homotopy Theorem 3.5 from *Fibrations II*, which state that if $f \simeq g : X \to Y$ are homotopic maps and $E \to Y$ is a fibration, then the pullback spaces $f^*E \to X$ and $g^*E \to X$ are homotopy equivalent over X. We apply this to the pullbacks defining the mapping path spaces.

Proposition 2.4 *Fix spaces X, Y. If $f \simeq g : X \to Y$ are homotopic maps, then there is a fibrewise homotopy equivalence $W_f \simeq_X W_g$ over X.* $\blacksquare$

The conclusion is that the construction of the mapping path space W_f makes sense on the level of the homotopy category. If $[f] : X \to Y$ is a homotopy class of maps, then as long as we are willing to replace X with an equivalent object, we can always choose a representative for $[f]$ which is a fibration. The proposition says that this makes complete sense if we stay within the homotopy category.

2.2 Functorality

Assume given a square with homotopy

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"'] & Y'. \end{tikzcd} \tag{2.16}$$

We use this to define a map

$$\theta_F = \theta(\alpha, F, \beta) : W_f \to W_{f'} \tag{2.17}$$

by setting

$$\theta_F(x, l) = (\alpha(x), F(x) + \beta l), \qquad (x, l) \in W_f. \tag{2.18}$$

This is well-defined since $F_0 = f'\alpha$ and $F_1 = \beta f$. In the diagram

$$\begin{tikzcd} X \arrow[r, "s_f"] \arrow[d, "\alpha"] & W_f \arrow[r, "p_f"] \arrow[d, "\theta_F"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "s_{f'}"] & W_{f'} \arrow[r, "p_{f'}"] & Y' \end{tikzcd} \tag{2.19}$$

the left-hand square commutes up to homotopy and the right-hand square commutes strictly. With regards to the left-hand square we have

$$s_{f'}\alpha(x) = (\alpha(x), c_{\alpha(x)}), \qquad \theta_F s_f(x) = (\alpha(x), F(x) + c_{\beta f(x)}). \tag{2.20}$$

and we get a homotopy $L : s_{f'}\alpha \simeq \theta_F s_f$ by setting $L_s(x, l) = (x, L^{(1)}(x, l) + L^{(2)}(x, l))$, where $L^{(1)}(x, l), L^{(1)}(x, l)$ are the paths

$$L_s^{(1)}(x, t) = \begin{cases} f'(\alpha(x)) & 0 \le t \le 1 - s \\ F(x, t - (1 - s)) & 1 - s \le t \le 1 \end{cases} \qquad L_s^{(2)}(x, t) = \begin{cases} F\left(x, \frac{2}{2-s}t\right) & 0 \le t \le \frac{2-s}{2} \\ \beta f(x) & \frac{2-s}{2} \le t \le 1. \end{cases} \tag{2.21}$$

Lemma 2.5 *The homotopy class of $\theta_F : W_f \to W_{f'}$ depends only on the track homotopy class of F.*

Proof Let $\psi : F \sim F'$ be a track homotopy, considered as a map $X \times I \to Y'^I$ satisfying

- $\psi_0(x)(t) = F_t(x)$, $\psi_1(x)(t) = F'_t(x)$
- $\psi_s(x)(0) = f'(\alpha(x))$, $\psi_s(x)(1) = \beta(f(x))$

for $x \in X$, $s, t \in I$. Now consider the homotopy $\Psi : W_f \times I \to W_{f'}$ defined by

$$\Psi_s(x, l) = (x, \psi_s(x, l) + \beta l) \tag{2.22}$$

The conditions above show that this is well-defined and we check that it is a homotopy $\theta_F \simeq \theta_{F'}$. ∎

Now assume given a second square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\overset{F'}{\Rightarrow}"] & Y' \arrow[d, "\beta'"] \\ X'' \arrow[r, "f''"'] & Y'' \end{tikzcd} \tag{2.23}$$

and define $\theta_{F'} : W_{f'} \to W_{f''}$ as above.

Lemma 2.6 *There is a homotopy $\theta_{F'} \circ \theta_F \simeq \theta_{F'\Box F} : W_f \to W_{f''}$, where*

$$F'\Box F = F'\alpha + \beta' F :'' (\alpha'\alpha) \simeq (\beta'\beta) f. \tag{2.24}$$

Proof We have

$$\theta_{F'}\theta_F(x, l) = \big(\alpha'\alpha(x), F'\alpha(x) + (\beta' F(x) + \beta'\beta l)\big) \tag{2.25}$$

and

$$\theta_{F'\Box F}(x, l) = \big(\alpha'\alpha(x), (F'\alpha(x) + \beta' F(x)) + \beta'\beta l\big). \tag{2.26}$$

Cclearly these maps are related by a linear reparametrisation of the path coordinate. ∎

Define a category $Top(\to)$ as follows. The objects are maps $f : X \to Y$. A morphism from f to $f' : X' \to Y'$ is given by a homotopy commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\overset{[F]}{\Rightarrow}"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd} \tag{2.27}$$

which is equipped with a particular track homotopy class of homotopy $F : \beta f \simeq f'\alpha$. If $f'' : X'' \to Y''$ is a third map and

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\overset{[F']}{\Rightarrow}"] & Y' \arrow[d, "\beta"] \\ X'' \arrow[r, "f''"'] & Y'' \end{tikzcd} \tag{2.28}$$

represents a morphism $f' \to f''$, then we define the composite square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha'\alpha"'] \arrow[dr, phantom, "\overset{[F'\Box F]}{\Rightarrow}"] & Y \arrow[d, "\beta'\beta"] \\ X'' \arrow[r, "f''"'] & Y'' \end{tikzcd} \tag{2.29}$$

where $F'\Box F$ is as in (2.24). The track class of $F'\Box F$ depends only on the track classes of F, F' so this is well-defined. Putting Lemmas 2.5, 2.6 together we get the following.

Proposition 2.7 *The mapping path space defines a functor $Top(\to) \xrightarrow{f \mapsto W_f} hTop$.* ∎

3 The Homotopy Fibre of f

We now specialise to the case that $f : X \to Y$ is a pointed map between based spaces X, Y. All spaces, maps and homotopies will be based in this section, and *by fibration we will mean pointed fibration.* Recall that we give I the basepoint 1 and denote

$$PY = C_*(I, Y) = \{l : I \to Y \mid l(1) = *\}. \tag{3.1}$$

The evaluation map $e_0 : PY \to Y$ is a fibration and we showed in *Pointed Homotopy* Lemma 1.5 that PY is contractible.

Definition 1 *Define the* ***homotopy fibre*** *of a pointed map $f : X \to Y$ by means of the following pullback diagram in Top_**

$$\begin{tikzcd} F_f \arrow[r] \arrow[d, "i_f"'] \arrow[dr, phantom, "\ulcorner"] & PY \arrow[d, "e_0"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{3.2}$$

We denote by

$$i_f : F_f \to X \tag{3.3}$$

the canonical map. □

The homotopy fibre is defined with a special purpose in mind. The universal property of the pullback shows that maps $K \to F_f$ are in one-to-one correspondence with pairs consisting of *i*) a map $g : K \to X$, and *ii*) a null homotopy $H : fg \simeq *$. Our favoured model for the homotopy fibre is

$$F_f = \{(x, l) \in X \times Y^I \mid f(x) = l(0),\ l(1) = *\} \subseteq X \times Y^I \tag{3.4}$$

with i_f being the projection onto the first factor. We apply Proposition 1.5: i_f is a fibration and there is a homeomorphism

$$i_f^{-1}(*) \cong \Omega Y \tag{3.5}$$

between the fibre of i_f and the fibre of $e_0 : PY \to Y$.

Reasoning similarly to the last subsection we have the following very important fact.

Proposition 3.1 *Fix spaces X, Y. If $f \simeq g : X \to Y$ are homotopic maps, then there is a fibrewise homotopy equivalence $F_f \simeq_X F_g$ over X.* ∎

The point is that the homotopy type of the homotopy fibre of $f : X \to Y$ depends only on the homotopy class of f. Thus we are able to make sense of the construction at the level of the homotopy category. We would like to stress that really it is not the space F_f which is the homotopy fibre of f, but rather the entire pullback square (3.2) including F_f, the map i_f, and the canonical null homotopy $fi_f \simeq *$. Of course even this makes sense in light of Proposition 3.1.

Now recall the mapping path space W_f and the fibration $p_f : W_f \to Y$, $(x, l) \mapsto l(1)$. Notice that $p_f^{-1}(*) = F_f$. That is, the fibre of p_f is exactly the homotopy fibre of f and there is a strict fibration sequence of the form

$$F_f \to W_f \xrightarrow{p_f} Y. \tag{3.6}$$

There is also a factorisation of i_f as in the strictly commutative diagram

$$\begin{tikzcd} & F_f \arrow[dl] \arrow[dr, "i_f"] & \\ W_f \arrow[rr, "\pi_f"] & & X. \end{tikzcd} \tag{3.7}$$

Proposition 3.2 *Let $f : X \to Y$ be a pointed map. Then for any pointed space K the sequence*

$$[K, F_f] \xrightarrow{i_f} [K, X] \xrightarrow{f_*} [K, Y] \tag{3.8}$$

is exact in Set_*.

Proof Consider the following diagram

$$\begin{tikzcd} {[K,F_f]} \arrow[r] \arrow[d, equal] & {[K,W_f]} \arrow[r, "p_{f*}"] \arrow[d, "\pi_{f*}"] & {[K,Y]} \arrow[d, equal] \\ {[K,F_f]} \arrow[r, "i_{f*}"] & {[K,X]} \arrow[r, "f_*"] & {[K,Y]}. \end{tikzcd} \tag{3.9}$$

The left-hand square commutes due to (3.8) and the right-hand square commutes because $f\pi_f = (p_f s_f)\pi_f \simeq p_f$. By Proposition 1.6 the top row is exact, and since the homotopy equivalence π_f induces a bijection π_{f*} we get the proposition. ∎

We call the sequence

$$F_f \xrightarrow{i_f} X \xrightarrow{f} Y \tag{3.10}$$

a (short) **homotopy fibration sequence**. It is pointwise equivalent to a strict fibration sequence and the proposition shows that in many respects it behaves like one.

So what is the difference between a 'strict' and a 'homotopy' fibration sequence? As we now show, there is no difference if f is a fibration. Let $F = f^{-1}(*) \subseteq X$ and denote the inclusion $i : F \hookrightarrow X$. Since $fi = *$ we take the constant homotopy to get a canonical factorisation of i through the homotopy fibre i_f by a map

$$\varphi : F \to F_f. \tag{3.11}$$

Proposition 3.3 *If $f : X \to Y$ is a fibration between well-pointed space X, Y. Then φ is a homotopy equivalence.*

Proof This follows from Proposition 2.3. We know that W_f is homotopy equivalent to X over Y, and since the pullback functor is homotopical a choice of fibrewise homotopy equivalence $X \simeq W_f$ induces one between the fibres $F = f^{-1}(*)$ and $W_f = p_f^{-1}(*)$. For such an equivalence we choose the map $s_f : X \to W_f$, and then the map induced between fibres is exactly φ. The assumptions of well-pointedness are sufficient to guarantee that we do not leave the pointed category. ∎

Example 3.1 *Let $f : * \to Y$. Then $F_f \cong \Omega Y$ and there is a homotopy fibration sequence*

$$\Omega Y \to * \to Y. \tag{3.12}$$

On the other hand the strict fibre of f is a single point. □

Example 3.2 *Let $f = * : X \to Y$. Then*

$$F_f \cong X \times \Omega Y. \tag{3.13}$$

This can be seen directly, but an easier to get it is to use Lemma 4.1 below. □

Example 3.3 *If $f : X \to Y$ and $f' : X' \to Y'$ are given, then the homotopy fibre of $f \times f' : X \times X' \to Y \times Y'$ is homeomorphic to $F_f \times F_{f'}$. Note that $P(X \times X') \cong PX \times PX'$.* □

4 Fibre Sequences

In this section we will generalise Proposition 1.6 and produce for any map f a long fibration sequence. We continue to work exclusively in the pointed category.

Lemma 4.1 *Consider the following pair of diagrams*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(I)"] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "(II)"] & E \arrow[d] \\ B \arrow[r] & D \arrow[r] & F \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(III)"] & E \arrow[d] \\ B \arrow[r] & F \end{tikzcd} \tag{4.1}$$

where the horizontal maps in the right-hand diagram are the composites of those in the left-hand diagram. The following statements hold.

1) If squares (I) and (II) are pullbacks, then so is (III).

2) If squares (II) and (III) are pullbacks, then so is (I).

Proof This is a simple check of the universal properties. ∎

Fix a pointed map $f : X \to Y$. Set $i_0 = i_f : F_0 = F_f \to X$ and for $n \geq 1$ inductively write $i_n = i_{i_{n-1}} : F_n = F_{i_{n-1}} \to F_{n-1}$ for the homotopy fibre of i_{n-1}. Now consider the following diagram of iterated pullbacks

$$\begin{tikzcd} & F_3 \arrow[r, "i_3"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & F_2 \arrow[r] \arrow[d, "i_2"'] \arrow[dr, phantom, "\ulcorner"] & PF_0 \arrow[d] & \\ & PF_1 \arrow[r] & F_1 \arrow[r, "i_1"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & F_0 \arrow[r] \arrow[d, "i_0"'] \arrow[dr, phantom, "\ulcorner"] & PY \arrow[d] \\ & & PX \arrow[r] & X \arrow[r, "f"] & Y. \end{tikzcd} \tag{4.2}$$

Each subsequent square defines the homototopy fibre of the map generated by the previous square. Notice that the left-hand squares are twisted. This is essentially the cause of the appearance of a -1 sign below. We apply Lemma 4.1 to get pullback squares

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & PY \arrow[d] \\ PX \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} F_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & PF_0 \arrow[d] \\ PX \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} F_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & PF_0 \arrow[d] \\ PF_1 \arrow[r] & F_0. \end{tikzcd} \tag{4.3}$$

Since $PY \to Y$ is a fibration, so is $F_1 \to PX$ and the fibres of both maps are homeomorphic to ΩY. Moreover, because PX is contractible the inclusion $\Omega Y \hookrightarrow F_1$ is a homotopy equivalence. Similarly statements apply to the other two pullbacks. In the middle square the map $F_2 \to PF_0$ is a fibration with fibre ΩX and the inclusion $\Omega X \hookrightarrow F_{i_2}$ is a homotopy equivalence, and in the right-hand square $F_3 \to PF_1$ is a fibration whose fibre inclusion $\Omega F_0 \hookrightarrow F_3$ is a homotopy equivalence.

Lemma 4.2 *The following diagram commutes up to homotopy*

$$\begin{tikzcd} \Omega X \arrow[r, "\simeq"] \arrow[d, "-\Omega f"'] & F_2 \arrow[d, "i_2"] \\ \Omega Y \arrow[r, "\simeq"] & F_1 \end{tikzcd} \tag{4.4}$$

where the horizontal arrows are the fibre inclusions described above.

Proof We'll record all the details, making reference to (4.3) for ease. We have

$$F_1 \cong \{(k, l) \in PX \times PY \mid f(k(0)) = l(0)\} \tag{4.5}$$
$$F_2 \cong \{(k, m) \in PX \times PF_0 \mid k(0) = i_0(m(0))\} \tag{4.6}$$

and the map i_2 is given by

$$i_2(k,m) = (k, i_0 \circ m). \tag{4.7}$$

The inclusions are the maps

$$\Omega Y \xrightarrow{\simeq} F_{i_0} \qquad l \mapsto (*, l) \tag{4.8}$$

$$\Omega X \xrightarrow{\simeq} F_{i_1} \qquad k \mapsto (k, *) \tag{4.9}$$

which as discussed above are homotopy equivalences. We claim that the maps

$$F_1 \xrightarrow{\simeq} \Omega Y \qquad (k,l) \mapsto -f \circ k + l \tag{4.10}$$

$$F_2 \xrightarrow{\simeq} \Omega X \qquad (k,m) \mapsto -i_0 \circ m + k \tag{4.11}$$

are their respective inverses. To check this it will suffice to show that they are right homotopy inverses, since we know a priori that (4.8), (4.9) are homotopy equivalences. But this is easy to see, since the composite $\Omega Y \to F_1 \to \Omega Y$ is the map $l \mapsto - * + l$, which is clearly homotopic to the identity. Similarly, the composite $\Omega X \to F_2 \to \Omega X$ is the map $k \mapsto - * + k$ which is homotopic to the identity.

Finally we check that the composite

$$\Omega X \xrightarrow{\simeq} F_2 \xrightarrow{i_2} F_1 \xrightarrow{\simeq} \Omega Y \tag{4.12}$$

is the map

$$l \mapsto -\Omega f(l) \tag{4.13}$$

which is exactly what was claimed. ∎

Lemma 4.3 *The following diagram commutes up to homotopy*

$$\begin{tikzcd} \Omega F_0 \arrow[r, "\simeq"] \arrow[d, "-\Omega i_0"'] & F_3 \arrow[d, "i_3"] \\ \Omega X \arrow[r, "\simeq"] & F_2 \end{tikzcd} \tag{4.14}$$

where the horizontal arrows are the fibre inclusions described above.

Proof The proof is similar to the last lemma so we will be brief. We have

$$F_3 \cong \{(m,n) \in PF_1 \times PF_0 \mid i_1(m(0)) = n(0)\} \tag{4.15}$$

and $i_3(m,n) = (m(0), n)$. With reference to (4.11) the composite

$$\Omega F_0 \xrightarrow{\simeq} F_3 \xrightarrow{i_3} F_2 \xrightarrow{\simeq} \Omega X \tag{4.16}$$

is the map $n \mapsto -\Omega i_0(n) + *$. ∎

Next we show that ΩF_0 identifies canonically with the homotopy fibre of $-\Omega f$. For this we need the switch of variables homeomorphism

$$P\Omega Y = C_*(I, C_*(S^1, Y)) \cong C_*(I \wedge S^1, Y) \cong C_*(S^1, C_*(I, Y)) \cong \Omega PY. \tag{4.17}$$

Lemma 4.4 *The pair of $-\Omega i_f$ and the switch map* (4.17) *induce a homeomorphism*

$$\Omega F_0 \cong F_{-\Omega f} \tag{4.18}$$

Proof Since the loop functor is a right adjoint it preserves pullbacks, meaning that

$$\begin{tikzcd}
\Omega F_0 \arrow[r] \arrow[d, "\Omega i_0"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Omega PY \arrow[d] \\
\Omega X \arrow[r, "\Omega f"] & \Omega Y
\end{tikzcd} \tag{4.19}$$

is a pullback. This square is related to the pullback defining $F_{-\Omega f}$ by the induced map in the diagram

$$\begin{tikzcd}
\Omega F_0 \arrow[rr] \arrow[dr, dashed] \arrow[dd, "\Omega i_0"'] & & \Omega PY \arrow[d, "\cong"] \\
 & F_{-\Omega f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & P\Omega Y \arrow[d] \\
\Omega X \arrow[r, "-1", "\cong"'] & \Omega X \arrow[r, "-\Omega f"] & \Omega Y.
\end{tikzcd} \tag{4.20}$$

We have $(-1) \circ \Omega i_0 = -\Omega i_0$ by definition and $(-\Omega f) \circ (-1) = -(-\Omega f) = \Omega f$ since Ωf is a loop map. Thus (4.19) and (4.20) are the same pullback. ∎

Define the fibration **connecting map** $\delta_f : \Omega Y \to F_0$ to be the composite

$$\delta_f : \Omega Y \xrightarrow{\simeq} F_1 \xrightarrow{i_1} F_0, \qquad l \mapsto (*, l). \tag{4.21}$$

where the first map is (4.8). Then we have the following diagram of spaces and maps

$$\begin{tikzcd}
\ldots \arrow[r] & \Omega F_0 \arrow[r, "-\Omega i_0"] \arrow[d, "\simeq"'] & \Omega X \arrow[r, "-\Omega f"] \arrow[d, "\simeq"'] & \Omega Y \arrow[d, "\simeq"'] \arrow[dr, dashed, "\delta_f"] & & & \\
\ldots \arrow[r] & F_3 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r, "i_0"] & X \arrow[r, "f"] & Y.
\end{tikzcd} \tag{4.22}$$

Each pair of composable maps in the bottom row consists of a map and its homotopy fibre and the vertical arrows are homotopy equivalences. According to Lemma 4.4 the two maps on the top row form a homotopy fibration sequence, so by replacing f, i_0 with $-\Omega f, -\Omega i_0$ we can interate the construction and so extend the diagram infinitely to the left. We arrive at the sequence of spaces and maps

$$\ldots \to \Omega^n F_f \xrightarrow{(-1)^n \Omega^n i_f} \Omega^n X \xrightarrow{(-1)^n \Omega^n f} \Omega^n Y \to \ldots \to \Omega F_f \xrightarrow{-\Omega i_f} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\delta} F_f \xrightarrow{i_f} X \xrightarrow{f} Y \tag{4.23}$$

which we call the **long fibration sequence** associated with f. Each pair of subsequent maps is pointwise equivalent to a short homotopy fibration sequence, and thus pointwise equivalent to a strict fibration sequence.

Theorem 4.5 *Let $f : X \to Y$ be a pointed map. Then for any space K the sequence*

$$\ldots \longrightarrow [K, \Omega X] \xrightarrow{\Omega f_*} [K, \Omega Y] \xrightarrow{\delta_{f*}} [K, F_f] \xrightarrow{i_{f*}} [K, X] \xrightarrow{f_*} [K, Y]$$

$$\ldots \longrightarrow [K, \Omega^{n+1} Y] \xrightarrow{\Omega^n \delta_{f*}} [K, \Omega^n F_f] \xrightarrow{\Omega^n i_{f*}} [K, \Omega^n X] \xrightarrow{\Omega^n f_*} [K, \Omega^n Y] \longrightarrow \ldots \tag{4.24}$$

is exact. Here the first three terms on the right are exact as pointed sets, the next three terms exact as groups, and all subsequent terms exact as abelian groups.

Proof Exactness of the first three terms was verified in Proposition 3.2. The same analysis applies to any term obtained by looping them, since the conclusion above was that

$$\Omega^n F_f \xrightarrow{(-1)^n \Omega^n i_f} \Omega^n X \xrightarrow{(-1)^n \Omega f} \Omega^n Y \tag{4.25}$$

is a homotopy fibration sequence. The appearance of the (-1) signs is inconsequential to the exactness of (4.24). For instance if $\alpha \in [K, \Omega X]$, then

$$-\Omega f \circ \alpha \simeq * \qquad \Leftrightarrow \qquad \Omega f \circ \alpha \simeq *. \tag{4.26}$$

It remains to show exactness at the sets involving $\Omega^n F_f$. Before looping this follows from the construction of (4.23), since Lemmas 4.2 and 4.3 give commutative diagrams

$$\begin{tikzcd}
{[K, \Omega X]} \arrow[r, "\delta_{f*}"] \arrow[d, "\cong"] & {[K, F_f]} \arrow[r, "i_{f*}"] \arrow[d, equal] & {[K, X]} \arrow[d, equal] \\
{[K, F_1]} \arrow[r, "i_{1*}"] & {[K, F_0]} \arrow[r, "i_{0*}"] & {[K, X].}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[K, \Omega Y]} \arrow[r, "-\Omega f_*"] \arrow[d, "\cong"] & {[K, \Omega X]} \arrow[r, "\delta_{f*}"] \arrow[d, "\cong"] & {[K, F_f]} \arrow[d, equal] \\
{[K, F_{i_2}]} \arrow[r, "i_{i_1 *}"] & {[K, F_{i_1}]} \arrow[r, "i_{0*}"] & {[K, F_0].}
\end{tikzcd} \tag{4.27}$$

with exact bottom rows and bijective vertical arrows. After looping the same diagrams commute and the bottom rows remain exact.

It remains to comment on the group structures. We know from our analysis of H-spaces that the homotopy sets of the form $[K, \Omega Q]$ are groups, and are abelian if $Q \simeq \Omega Q'$. Since the loop maps that appear in (4.23) induce group homomorphisms we find that the exactness in the sense of pointed sets improves to the familar exactness for groups and abelian groups. ∎

Now assume that $p : E \to B$ is a pointed fibration between well-pointed spaces E, B. With Proposition 3.3 in mind we ask how (4.23) changes if f is replaced by p. The canonical map $F = p^{-1}(*) \to F_p$ from the strict fibre of p to its homotopy fibre is a homotopy equivalence. We define

$$\delta : \Omega B \to F_p \tag{4.28}$$

to make the following diagram commute up to homotopy

$$\begin{tikzcd}
 & & F \arrow[r, "i"] \arrow[d, "\simeq"] & E \arrow[r, "p"] \arrow[d, equal] & B \arrow[d, equal] \\
\ldots \Omega E \arrow[r, "-\Omega p_*"] & \Omega B \arrow[r] \arrow[ur, dashed, "\delta"] & F_p \arrow[r] & E \arrow[r] & B.
\end{tikzcd} \tag{4.29}$$

The sequence we get in this case now looks like

$$\ldots \to \Omega^n F \xrightarrow{(-1)^n\Omega^n i} \Omega^n E \xrightarrow{(-1)^n\Omega^n p} \Omega^n B \to \ldots \to \Omega F \xrightarrow{-\Omega i} \Omega E \xrightarrow{-\Omega p} \Omega B \xrightarrow{\delta} F \xrightarrow{i} E \xrightarrow{p} B \tag{4.30}$$

in which each pair of composable arrows forms either a homotopy fibration sequence or a strict fibration sequence.

Corollary 4.6 *Let $p : E \to B$ be a pointed fibration with fibre F and let K be a pointed space. Assume that either E, B are well-pointed or that K is. Then the sequence*

$$\ldots \longrightarrow [K, \Omega E] \xrightarrow{\Omega p_*} [K, \Omega B] \xrightarrow{\delta_*} [K, F] \xrightarrow{i_*} [K, E] \xrightarrow{p_*} [K, B]$$

$$\ldots \longrightarrow [K, \Omega^{n+1} B] \xrightarrow{\Omega^n \delta_*} [K, \Omega^n F] \xrightarrow{\Omega^n i_*} [K, \Omega^n E] \xrightarrow{\Omega^n p_*} [K, \Omega^n B] \longrightarrow \ldots \tag{4.31}$$

is exact in the same sense of Theorem 4.5.

Proof In the case that E, B are well-pointed the statement follows from 4.5 and the construction of (4.30). If E, F are not well-pointed, then the comparison map $F \to F_f$ may fail to be a *pointed* homotopy equivalence. However, it is at least a free homotopy equivalence, so in the case that K is well-pointed it still induces a bijection $[K, F] \xrightarrow{\cong} [K, F_f]$. With this observation we can follow the proof of 4.5 with only minor changes. ∎

As a special case of the corollary we take $K = S^0$ and get what we call the **long exact homotopy sequence** of a fibration.

Corollary 4.7 *Let $p : E \to B$ be a pointed fibration with fibre F. Then there is a long exact sequence of homotopy groups*

$$\ldots \to \pi_{n+1} B \to \pi_n F \xrightarrow{i_*} \pi_n E \xrightarrow{p_*} \pi_n B \xrightarrow{\Delta} \pi_{n-1} F \to \ldots \tag{4.32}$$

which ends as the sequence of pointed sets

$$\ldots \to \pi_1 B \to \pi_0 F \to \pi_0 E \to \pi_0 B. \tag{4.33}$$

Proof We make use of the suspension-loop adjunction

$$[S^0, \Omega^n X] \cong [\Sigma^n S^0, X] \cong \pi_n X. \tag{4.34}$$

By the work in §3 of *H-Spaces I* we know that the group structure on the homotopy set is the same whether defined using the loop or suspension coordinates. In particular the exactness of (4.24) is the same as the exactness stated in Theorem 4.5. The boundary map Δ is obtained as the composite

$$\pi_n B \cong \pi_{n-1} \Omega B \xrightarrow{\delta_*} \pi_{n-1} F \tag{4.35}$$

where δ is as in (4.28). The first map here is the adjunction homomorphism, so Δ is a homomorphism. ∎

Of course there is a statement lying between 4.5 and 4.7. Namely that for any pointed map $f : X \to Y$ there is a long exact sequence of homotopy groups

$$\ldots \to \pi_{n+1}Y \to \pi_n F_f \xrightarrow{i_{f*}} \pi_n X \xrightarrow{f_*} \pi_n Y \xrightarrow{\Delta} \pi_{n-1}F_f \to \ldots \tag{4.36}$$

This makes the homotopy fibre F_f amenable to study by homotopical means. Compare this to the homotopy cofiber C_f, which is more understandable using cohomological methods.

5 Examples

5.1 The Circle

In *Fibrations III* we saw that covering projections are fibrations. Thus we take the universal covering space of the circle and get a fibration sequence

$$\mathbb{Z} \to \mathbb{R} \xrightarrow{p} S^1 \tag{5.1}$$

where

$$p(t) = \exp(2\pi i \cdot t). \tag{5.2}$$

Now in the long exact sequence homotopy sequence

$$\ldots \to \pi_k\mathbb{R} \xrightarrow{p_*} \pi_k S^1 \to \pi_{k-1}\mathbb{Z} \to \ldots \tag{5.3}$$

all homotopy groups of the contractible space $\mathbb{R}$ disappear, as do all homotopy groups of the discrete space $\mathbb{Z}$. Thus exactness gives us

$$\pi_k S^1 = 0, \qquad \forall k \geq 2. \tag{5.4}$$

All that remains of the sequence (5.3) is its tail end

$$1 \to \pi_1 S^1 \to \pi_0\mathbb{Z} = \mathbb{Z} \to *. \tag{5.5}$$

which retains exactness as pointed sets. Of course we calculated last lecture that

$$\pi_1 S^1 \cong \mathbb{Z} \tag{5.6}$$

but it's not difficult to solve the extension problem (5.5) directly. In any case, the fact that S^1 has exactly one non-vanishing homotopy group makes it a fairly special space. For example it's known that for each $n \geq 2$, $\pi_* S^n$ is non-zero is arbitrarily high degrees. We will explore some consequences of the calculation (5.4) in the examples below.

Here is another take on the above results. One upshot of our approach to fibration sequences is that we are often able to make much more precise statements that just perform algebraic computations. For example consider the pullback diagram defining the homotopy fibre of p

$$\begin{tikzcd}
 & \Omega S^1 \arrow[d, "\simeq"'] & \\
\mathbb{Z} \arrow[r, "\simeq"] \arrow[dr, bend right] & F_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & PS^1 \arrow[d] \\
 & \mathbb{R} \arrow[r] & S^1.
\end{tikzcd} \tag{5.7}$$

Here the induced map $\mathbb{Z} \xrightarrow{\simeq} F_p$ is a homotopy equivalence, as is the fibre inclusion $\Omega S^1 \to F_p$. This gives us an actual homotopy equivalence

$$\Omega S^1 \simeq \mathbb{Z} \tag{5.8}$$

and in particular a much stronger statement than the isomorphisms (5.4), (5.6).

To complete the picture let us make (5.8) a little more concrete. For each $s \in \mathbb{R}$ let $\alpha_s : I \to S^1$ be the path $\alpha_s(t) = \exp(2\pi i \cdot st)$. Then we check that the map

$$W_p \to \Omega S^1, \qquad (s, l) \mapsto \alpha_s + l \tag{5.9}$$

is the homotopy inverse to the vertical map in (5.7). Composing this with the horizontal map in the same diagram we get a map

$$\mathbb{Z} \to \Omega S^1, \qquad n \mapsto \alpha_n. \tag{5.10}$$

which induces (5.8). It is amusing to think that when we first learned to classify self maps of S^1 by their degrees, really we were really constructing a homotopy equivalence between ΩS^1 and the discrete integers. $\square$

5.2 Projective Spaces

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and write $d = d_{\mathbb{K}}$ for the dimension of $\mathbb{K}$ as a real vector space. Note that $d_{\mathbb{R}} = 1, d_{\mathbb{C}} = 2$ and $d_{\mathbb{H}} = 4$. To fix conventions for the noncommuative quaternions we consider $\mathbb{K}^n$ as a *right* $\mathbb{K}$-vector space. We write $|\lambda|$ for the modulus of $\lambda \in \mathbb{K}$ and $\|u\|$ for the norm of $u = (u_0, \ldots, u_n) \in \mathbb{K}^{n+1}$. We write $S(\mathbb{K}) = \{z \in \mathbb{K} \mid |z| = 1\}$ for the unit sphere in $\mathbb{K}$. In particular

$$S(\mathbb{R}) = S^0, \qquad S(\mathbb{C}) = S^1, \qquad S(\mathbb{H}) = S^3 \tag{5.11}$$

and each has the structure of a compact Lie group. We identify the unit sphere in $\mathbb{K}^{n+1}$ with $S^{d(n+1)-1}$. Then $S(\mathbb{K})$ acts on $S^{d(n+1)-1}$ from the right and we define the **$\mathbb{K}$-projective n-space** $\mathbb{K}P^n$ to be the set of cosets of this action

$$\mathbb{K}P^n = S^{d(n+1)-1}/S(\mathbb{K}). \tag{5.12}$$

We denote the elements of $\mathbb{K}P^n$ with square brackets and write

$$\gamma_n = \gamma_n^{\mathbb{K}} : S^{d(n+1)-1} \to \mathbb{K}P^n \tag{5.13}$$

for the quotient projection. The inclusion $\mathbb{K}^n \hookrightarrow \mathbb{K}^{n+1}$ as the first n non-zero coordinates induces inclusions $S^{dn-1} \hookrightarrow S^{d(n+1)-1}$ and $\mathbb{K}P^{n-1} \hookrightarrow \mathbb{K}P^n$.

Proposition 5.1 *For each $n \geq 1$ the space $\mathbb{K}P^n$ may be obtained from $\mathbb{K}P^{n-1}$ by attaching a d-cell along γ_{n-1}. In particular $\mathbb{K}P^n$ is dn-dimensional CW complex with one cell in each dimension $\leq dn$ congruent to d.*

Proof We view $S^{dn-1} \subseteq D^{dn} \subseteq \mathbb{K}^n$ and define $\Gamma : D^{dn} \to \mathbb{K}P^n$ by

$$\Gamma(u) = \left[u, \sqrt{1 - \|u\|}\right], \qquad u \in D^{dn} \subseteq \mathbb{K}^n. \tag{5.14}$$

Then $\Gamma|_{S^{dn-1}} = \gamma_{n-1}$ and it is easily checked that Γ gives a relative homeomorphism $(D^{dn}, S^{dn-1}) \cong (\mathbb{K}P^n, \mathbb{K}P^{n-1})$. The last statements of the proposition follow by induction. ∎

Note that $\mathbb{K}P^0$ is a single point.

Corollary 5.2 *There is a homeomorphism $\mathbb{K}P^1 \cong S^d$.* $\blacksquare$

The homeomorphism follows from the cell structure described in 5.1. We get an explicit map by viewing S^d as the one-point compactification of $\mathbb{K}$ and sending

$$[u_0, u_1] \mapsto \begin{cases} u_0 \cdot u_1^{-1} & u_1 \neq 0 \\ \infty & u_1 = 0. \end{cases} \tag{5.15}$$

Proposition 5.3 *For each $n \geq 0$ the projection $\gamma_n : S^{d(n+1)-1} \to \mathbb{K}P^n$ is a locally trivial fibration with fibre $S(\mathbb{K})$.*

Proof Since $\mathbb{K}P^n$ is a CW complex it will suffice to show that γ_n is locally trivial. For each $i = 0, \ldots, n$ write

$$U_i = \{[u_0, \ldots, u_n] \in \mathbb{K}P^n \mid u_i \neq 0\}. \tag{5.16}$$

Then $U_0, \ldots, U_n$ is an open cover of $\mathbb{K}P^n$ and we claim γ_n is trivial over each U_i. Indeed, define $\rho_i : U_i \times S(\mathbb{K}) \to \gamma_n^{-1}(U_i)$ by

$$\rho_i([u_0, \ldots, u_n], \lambda) = \lambda \frac{|u_i|}{u_i}(u_0, \ldots, u_n) \tag{5.17}$$

and $\theta_i : \gamma_n^{-1}(U_i) \to U_i \times S(\mathbb{K})$ by

$$\theta_i(u_0, \ldots, u_n) = \left([u_0, \ldots, u_n], \frac{u_i}{|u_i|}\right). \tag{5.18}$$

Then we check easily that ρ_i and θ_i are inverse homeomorphisms over U_i. $\blacksquare$

We study the three cases individually.

The Real Case: When $\mathbb{K} = \mathbb{R}$ we have fibre sequences

$$S^0 \to S^n \xrightarrow{\gamma_n^{\mathbb{R}}} \mathbb{R}P^n. \tag{5.19}$$

Thus $\gamma_n^{\mathbb{R}}$ is a two-sheeted covering projection for each n. When $n = 1$ it is the degree 2 map $S^1 \xrightarrow{2} S^1$. In other cases it is the universal covering space. The long exact sequence of homotopy groups 4.7 gives us for each $n \geq 2$ that

$$\pi_1 \mathbb{R}P^n \cong \mathbb{Z}_2 \tag{5.20}$$

and says that the projection $\gamma_n^{\mathbb{R}}$ induces isomorphisms

$$\pi_k \mathbb{R}P^n \cong \pi_k S^n, \qquad \forall k \geq 2. \tag{5.21}$$

We make use of the diversion on functorality. Each square

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S^{n+1} \arrow[d] \\ \mathbb{R}P^n \arrow[r, "i_n"] & \mathbb{R}P^{n+1} \end{tikzcd} \tag{5.22}$$

is by inspection a pullback. In particular the induced map of vertical fibres is a homeomorphism. The homotopy sequences are natural with respect to maps of fibrations (see §2.2 for the most general setup), so for $n \geq 1$ (5.22) gives us a diagram

$$\begin{tikzcd} \pi_1 S^n \arrow[r, "\gamma^{\mathbb{R}}_{n*}"] \arrow[d] & \pi_1 \mathbb{R}P^n \arrow[r] \arrow[d, "i_{n*}"'] & \pi_0 S^0 \cong \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \pi_1 \mathbb{R}P^{n+1} \arrow[r] & \pi_0 S^0 \arrow[r] & 0 \end{tikzcd} \tag{5.23}$$

with exact rows. We conclude from exactness that the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ induces an isomorphism on π_1 whenever $n \geq 2$, and when $n = 1$ a surjection $\pi_1 \mathbb{R}P^1 \to \pi_2 \mathbb{R}P^2$. In particular we get a canonical generator for $\pi_1 \mathbb{R}P^n \cong \mathbb{Z}_2$ as the inclusion

$$S^1 \cong \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n. \tag{5.24}$$

The Complex Case: When $\mathbb{K} = \mathbb{C}$ we have fibre sequences

$$S^1 \to S^{2n+1} \xrightarrow{\gamma^{\mathbb{C}}_n} \mathbb{C}P^n. \tag{5.25}$$

Making use of Example 5.1 we find

$$\pi_2 \mathbb{C}P^n \cong \pi_1 S^1 \cong \mathbb{Z} \tag{5.26}$$

and that

$$\pi_k \mathbb{C}P^n \cong \pi_k S^{2n+1}, \qquad \forall k \geq 3. \tag{5.27}$$

Again these isomorphisms are induced by the quotient projection $\gamma^{\mathbb{C}}_n$.

The special case $n = 1$ gives us the **complex Hopf fibration**

$$S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2. \tag{5.28}$$

where we write $\eta = \gamma^{\mathbb{C}}_1$. We saw in the exercise sheets that η generates a free $\mathbb{Z}$ summand in $\pi_3 S^2$. Equation (5.27) tells us that there is nothing else in this group. It also tells us that η induces isomorphisms

$$\eta_* : \pi_k S^3 \xrightarrow{\cong} \pi_k S^2, \qquad \forall k \geq 3. \tag{5.29}$$

We will state a much stronger result than this shortly.

The Quaternionic Case: In the quaternionic case we have fibre sequences

$$S^3 \to S^{4n+3} \xrightarrow{\gamma_n^{\mathbb{H}}} \mathbb{H}P^n. \tag{5.30}$$

The results differ from (5.20), (5.26) since the higher homotopy groups of S^3 are nontrivial. We compute

$$\pi_k \mathbb{H}P^n \cong \pi_{k-1} S^3, \qquad \forall\ 0 \leq k \leq 4n-3 \tag{5.31}$$

and find in particular that

$$\pi_4 \mathbb{H}P^n \cong \mathbb{Z}. \tag{5.32}$$

We compute in the next example that in general we have

$$\pi_k \mathbb{H}P^n \cong \pi_k S^{4n+3} \oplus \pi_{k-1} S^3 \tag{5.33}$$

(compare (5.27), (5.21)).

The special case $n = 1$ gives us the **quaternionic Hopf fibration**

$$S^3 \hookrightarrow S^7 \xrightarrow{\nu} S^4 \tag{5.34}$$

where we write $\nu = \gamma_1^{\mathbb{H}}$. By (5.33) the element ν generates a free $\mathbb{Z}$ summand in $\pi_7 S^4$, but in contrast to the real and complex cases this group also contains a copy of the non-trivial $\pi_6 S^3 \cong \mathbb{Z}_{12}$.

5.3 Loop Spaces of Projetive Spaces

Return to the general case and consider the long fibration sequence

$$\ldots \to \Omega S^{dn-1} \to \Omega \mathbb{K}P^n \to S^{d-1} \xrightarrow{i} S^{dn-1} \xrightarrow{\gamma_n} \mathbb{K}P^n. \tag{5.35}$$

Since $\pi_{d-1} S^{dn-1} = 0$ the fibre inclusion i is null-homotopic. In particular $\Omega \mathbb{K}P^n$ is homotopy equivalent to the fibre of the constant map $S^{d-1} \to S^{dn-1}$. Thus following Example 3.2 we can find a homotopy equivalence

$$\Omega \mathbb{K}P^n \simeq S^{d-1} \times \Omega S^{dn-1}. \tag{5.36}$$

Of course this returns all the results of the previous example since

$$\pi_k \mathbb{K}P^n \cong \pi_{k-1}(\Omega \mathbb{K}P^n) \cong \pi_{k-1}(S^{d-1} \times \Omega S^{dn-1}) \cong \pi_{k-1} S^{d-1} \oplus \pi_k S^{dn-1}. \tag{5.37}$$

We would like to stress that (5.36) is a much stronger result than its corollary (5.37). From our point of view having topological information is always preferable to having weaker algebraic information. You will notice that (5.36) required minimal work to obtain. Our methods have been tailored to obtaining these stronger results.

5.4 Delooping the Unit Sphere $S(\mathbb{K})$

Start with the fibration sequence

$$S^{d-1} \xrightarrow{i} S^{dn-1} \xrightarrow{\gamma_n} \mathbb{K}P^n \tag{5.38}$$

and consider the limit as $n \to \infty$. The infinite $\mathbb{K}$-projective space $\mathbb{K}P^\infty$ is formed as the union $\bigcup_{n\geq 0} \mathbb{K}P^n$ and given the weak topology with respect to the $\mathbb{K}P^n$. It is an infinite-dimensional CW complex with a single cell in each dimension congruent to d. Similarly we let $S^\infty = \bigcup_{n\geq 0} S^{dn-1}$ be the infinite dimensional sphere given the weak topology with respect to the finite dimensional spheres. Then S^∞ is a CW complex and the maps γ_n induce a map

$$\gamma_\infty : S^\infty \to \mathbb{K}P^\infty. \tag{5.39}$$

We check γ_∞ is trivial over each of the sets $U_i = \{[u] \in \mathbb{K}P^\infty \mid u_i \neq 0\}$ and conclude from the fact that $\mathbb{K}P^\infty$ is paracompact that γ_∞ is a fibration.

Lemma 5.4 $S^\infty \simeq *$

Proof Since S^∞ is a CW complex it is well-pointed. Thus to conclude that S^∞ is pointed contractible it will suffice to show that it is freely contractible. To this end we use the family of continuous maps

$$\begin{aligned} Sh_n : S^{dn-1} &\to S^{d(n+1)-1} \\ (u_1, \dots, u_n) &\mapsto (0, u_1, \dots, u_n) \end{aligned} \tag{5.40}$$

to induce a continuous shift map

$$Sh : S^\infty \to S^\infty. \tag{5.41}$$

The point $e_1 = (1, 0, \dots, 0, \dots)$ does not lie in the image of Sh, so by corestriction we can view Sh as a map $S^\infty \to S^\infty \setminus \{e_1\}$.

Now, if $u \in S^\infty$, then u is contained in S^{dn-1} for some n and we can check that the straight line in $\mathbb{K}^{n+1}$ between u and $Sh_n(u)$ does not pass through the origin. We conclude from this that the map

$$\begin{aligned} S^\infty \times I &\to S^\infty \\ (x, t) &\mapsto \frac{(1-t)x + tSh(x)}{\|(1-t)x + tSh(x)\|} \end{aligned} \tag{5.42}$$

is well-defined and continuous. It is a homotopy $id_{S^\infty} \simeq Sh$, where the codomain of Sh is $S^\infty \setminus \{e_1\}$. This latter space is homeomorphic to $\mathbb{R}^\infty$ in the weak topology and so is contractible. In this way we get a free contraction of S^∞. ∎

The first application of the lemma comes from the obvious extension of (5.36). Namely we have a fibration sequence

$$\dots \to \Omega S^\infty \to \Omega \mathbb{K}P^\infty \xrightarrow{\delta} S^{d-1} \xrightarrow{i} S^\infty \xrightarrow{\gamma_\infty^{\mathbb{K}}} \mathbb{K}P^\infty \tag{5.43}$$

in which $S^\infty \simeq *$. The connecting map δ is pointwise equivalent to the inclusion of a fibration with contractible base. In particular it induces a homotopy equivalence

$$S^{d-1} \simeq \Omega\mathbb{K}P^\infty. \tag{5.44}$$

This is quite fascinating. It seems counter intuitive that the loop space of the infinite complex $\mathbb{K}P^\infty$ should be homotopy equivalent to a complex with just one cell! The function space $\Omega\mathbb{K}P^\infty$ itself is huge. For contrast, the loop space ΩS^n has infinitely many cells if $n \geq 1$ and is infinite dimensional if $n \geq 2$.

Of course read in the other direction the equation shows that the spheres S^0, S^1 and S^3 have the homotopy types of loop spaces. We know that out of all the spheres, it is only S^0, S^1, S^3 and S^7 which admit H-space structures, and only S^0, S^1, S^3 which admit associative H-structures. It is actually no coincidence that these spheres are loop spaces, although this is not a connection we will not explore in these notes.

Now, (5.44) has another interesting consequence when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Namely we can compute that these infinite projective spaces have exactly one non-vanishing homotopy group

$$\pi_k\mathbb{R}P^\infty \cong \begin{cases} \mathbb{Z}_2 & k=1 \\ 0 & \text{otherwise} \end{cases} \qquad \pi_k\mathbb{C}P^\infty \cong \begin{cases} \mathbb{Z} & k=2 \\ 0 & \text{otherwise.} \end{cases} \tag{5.45}$$

Thus these spaces are examples of *Eilenberg-Mac Lane Spaces*. i.e. CW complexes having exactly one non-vanishing homotopy group. We can always construct such spaces by simply attaching cells to a suitable Moore space so as to kill all the relevant homotopy groups, but generally the spaces so constructed are not easy to understand. The infinite projective spaces $\mathbb{R}P^\infty, \mathbb{C}P^\infty$ are rare examples of Eilenberg-Mac Lane Spaces which carry some geometric significance.

References

[1] R. Kieboom, *Regular Fibrations Revisited*, Arch. Math., **45**, (1985), 68-73.